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Existence of Positive Periodic Solutions for a Class of Difference Equations with Several Deviating Arguments

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Abstract—In this paper, we employ the Mawhin continuation theorem to study the existence of positive periodic solutions for a kind of nonautonomous difference equation with several deviating arguments. Applying the general theorems established to several biomathematical models, we obtain some new results.

Keywords—Difference equation, Logistic equation, Existence, Positive periodic solution.

1. INTRODUCTION

For notation, given a, b are integers and $a < b$, we employ intervals to denote the discrete set such as $Z[a,b] = \{a, a+1, \dots, b\}$, $Z[a,b) = \{a, \dots, b-1\}$, $Z[a,\infty) = \{a, a+1, \dots\}$, etc. Let $T \in Z[1,\infty)$ be fixed.

The purpose of the present paper is to deal with the existence of positive T-periodic solutions for the general periodic logistic difference equations

$$\Delta x(t) = x(t)[a(t) - g(t, x(t-\tau_1(t)), \dots, x(t-\tau_n(t)))], \qquad t \in Z(-\infty,\infty), \tag{1.1}$$

where $\Delta x(t) = x(t+1) - x(t)$, $g : Z(-\infty,\infty) \times [0,\infty)^n \to [0,\infty)$, and $g(t, u_1, \dots, u_n)$ is continuous for $(u_1, \dots, u_n) \in [0,\infty)^n$. Here $a(t) : Z(-\infty,\infty) \to (0,\infty)$, $\tau_i(t) : Z(-\infty,\infty) \to Z(-\infty,\infty)$, $i = 1, \dots, n$, and $a(t) = a(t+T)$, $\tau_i(t) = \tau_i(t+T)$, $i = 1, \dots, n$, $g(t, u_1, u_2, \dots, u_n) = g(t+T, u_1, u_2, \dots, u_n)$.

It is well known that equation (1.1) includes many mathematical ecological difference logistic equations. For example, equation (1.1) includes a single species discrete periodic population model [1–7]

$$\Delta x(t) = a(t)x(t)\left[1 - \frac{x(t-\tau(t))}{K(t)}\right], \qquad t \in Z(-\infty,\infty), \tag{1.2}$$

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where $a(t) : Z(-\infty,\infty) \to (0,\infty)$, $\tau(t) : Z(-\infty,\infty) \to Z(-\infty,\infty)$, $K(t) : Z(-\infty,\infty) \to (0,\infty)$, and $a(t) = a(t+T)$, $K(t) = K(t+T)$, $\tau(t) = \tau(t+T)$; and

$$\Delta x(t) = x(t)\left[a(t) - \sum_{i=1}^{n} b_i(t)x(t-\tau_i(t))\right], \qquad t \in Z(-\infty,\infty), \tag{1.3}$$

where $a(t) : Z(-\infty,\infty) \to (0,\infty)$, $b_i(t) : Z(-\infty,\infty) \to (0,\infty)$, $\tau_i(t) : Z(-\infty,\infty) \to Z(-\infty,\infty)$, and $a(t) = a(t+T)$, $b_i(t) = b_i(t+T)$, $\tau_i(t) = \tau_i(t+T)$, $i = 1,\dots,n$.

Also, equation (1.1) includes

(i) a multiplicative delay periodic logistic difference equation [1–4,8,9]

$$\Delta x(t) = a(t)x(t)\left[1 - \prod_{i=1}^{n} \frac{x(t-\tau_i(t))}{K(t)}\right], \qquad t \in Z(-\infty,\infty), \tag{1.4}$$

where $a(t), K(t) : Z(-\infty,\infty) \to (0,\infty)$, $b_i(t) : Z(-\infty,\infty) \to (0,\infty)$, $\tau_i(t) : Z(-\infty,\infty) \to Z(-\infty,\infty)$, and $a(t) = a(t+T)$, $K(t) = K(t+T)$, $b_i(t) = b_i(t+T)$, $\tau_i(t) = \tau_i(t+T)$, $i = 1,\dots,n$;

(ii) a periodic Michaelis-Menton discrete model [2,10]

$$\Delta x(t) = a(t)x(t)\left[1 - \sum_{i=1}^{n} \frac{a_i(t)x(t-\tau_i(t))}{1 + c_i(t)x(t-\tau_i(t))}\right], \qquad t \in Z(-\infty,\infty), \tag{1.5}$$

where $a(t), a_i(t), c_i(t) : Z(-\infty,\infty) \to (0,\infty)$, $\tau_i(t) : Z(-\infty,\infty) \to Z(-\infty,\infty)$, and $a(t) = a(t+T),)$, $b_i(t) = b_i(t+T)$, $c_i(t) = c_i(t+T)$, $\tau_i(t) = \tau_i(t+T)$, $i = 1,\dots,n$.

During the last years, many authors have investigated the existence of positive periodic solutions for differential equations with several deviating arguments; for examples, see [1,2] and the references therein. As for the related continuous systems of (1.1), Li [2] has studied the existence of positive periodic solutions of (1.1) by applying the Mawhin continuation theorem, and Jiang and Wei [1] have also studied it by applying the Krasnoselskii fixed-point theorem in cones.

Clearly, systems (1.2)–(1.5) are special cases of (1.1). To the knowledge of the authors, there are very few works on the existence of positive periodic solutions for equation (1.1), even for equations (1.2)–(1.5). In [7], the authors have dealt with periodic solutions of a single species discrete population model with periodic harvest/stock; the model is the following:

$$x(t+1) = \mu x(t)\left[1 - \frac{x(t)}{K}\right] + b(t), \qquad t \in Z(-\infty,\infty), \tag{1.6}$$

where $\mu > 0$, $K > 0$, $b(t) : Z(-\infty,\infty) \to (-\infty,\infty)$, and $a(t) = a(t+T)$.

In this short note, we apply the Mawhin continuation theorem to establish a group of conditions to guarantee (1.1) to have positive periodic solutions. The conditions can be checked easily.

2. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

In this section, we shall study the existence of T-periodic solutions of equation (1.1). For the reader's convenience, we shall first summarize in the following a few concepts and results from [11] that will be basic for this section.

Let X, N be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{co}\dim \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q)$. It follows that $L \mid \operatorname{dom} L \cap \operatorname{Ker} P : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

In the proof of our existence theorem below, we will use the continuation theorem of Gaines and Mawhin [11, p. 40].

THEOREM A. CONTINUATION THEOREM. *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{dom} L \cap \bar{\Omega}$.

THEOREM 1. *Suppose the following.*

(H_1) *There exists $H > 0$ such that*

$$g(t, u_1, u_2, \ldots, u_n) > a(t), \qquad \text{for } t \in Z[0, T-1],$$

when $u_i \geq H$, $i = 1, \ldots, n$.

(H_2) *There exists $\varepsilon > 0$ ($\varepsilon < H$) such that*

$$g(t, u_1, u_2, \ldots, u_n) < a(t), \qquad \text{for } t \in Z[0, T-1],$$

when $0 < u_i \leq \varepsilon$, $i = 1, \ldots, n$.

Then equation (1.1) has at least one positive periodic solution of period T.

PROOF OF THEOREM 1. Let

$$X = Z = \{x(t) : Z(-\infty, \infty) \to R,\ x(t+T) = x(t)\} \tag{2.1}$$

with the norm $\|x\| = \max_{t \in Z[0,T-1]} |x(t)|$ for any $x \in X$ (or Z). Then X and Z are both Banach spaces when they are endowed with the norm $\|\cdot\|$, and $\dim X = \dim Z = T$. Let

$$(Nx)(t) := x(t)[a(t) - g(t, x(t-\tau_1(t)), \ldots, x(t-\tau_n(t)))],$$

for any $x \in X$.

$$\begin{aligned} Lx = \Delta x, \quad Px &= \frac{1}{T}\sum_{t=0}^{T-1} x(t), \qquad x \in X; \\ Qz &= \frac{1}{T}\sum_{t=0}^{T-1} z(t), \qquad z \in Z. \end{aligned} \tag{2.2}$$

Obviously,

$$\begin{aligned} \operatorname{Ker} L &= \{x \mid x \in X,\ x = h,\ h \in R\}, \\ \operatorname{Im} L &= \left\{ z \mid z \in Z,\ \sum_{t=0}^{T-1} z(t) = 0 \right\}, \end{aligned} \tag{2.3}$$

and

$$\dim \operatorname{Ker} L = 1 = \operatorname{co}\dim \operatorname{Im} L. \tag{2.4}$$

Since $\operatorname{Im} L$ is closed in Z, L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L. \tag{2.5}$$

Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ ($\operatorname{Dom} L = X$) is given by

$$K_P(z) = \sum_{s=0}^{t-1} z(s) - \frac{1}{T}\sum_{s=0}^{T-1}(T-s)z(s). \tag{2.6}$$

Thus, $QN : X \to Z$ is given by

$$QNx := \frac{1}{T}\sum_{t=0}^{T-1} x(t)[a(t) - g(t, x(t-\tau_1(t)), \dots, x(t-\tau_n(t))].$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Since $\dim X = T$, by using the Arzela-Ascoli theorem, it is not difficult to show that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$. The isomorphism J of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$ can be the identity mapping, since $\operatorname{Im} Q = \operatorname{Ker} L$.

In order to prove the existence of a positive T-periodic solution of equation (1.1), we need the following lemma.

LEMMA 1. *Let $x(t)$ be a positive T-periodic solution of (1.1). Then*

$$\min_{t \in Z[0,T-1]} x(t) \geq \sigma \|x\|,$$

where

$$\sigma = \left[\prod_{t=0}^{T-1}(1+a(t))\right]^{-1}.$$

PROOF. From (1.1), we have

$$x(t+1) = (a(t)+1)x(t) - x(t)g(t, x(t-\tau_1(t)), \dots, x(t-\tau_n(t))], \qquad x \in X. \tag{2.7}$$

Since $x(0) = x(T)$, we obtain that

$$x(t) = \sum_{t=0}^{T-1} G(t,s)x(s)g(s, y(s-\tau_1(s)), \dots, y(s-\tau_n(s))), \tag{2.8}$$

where

$$G(t,s) := \begin{cases} \dfrac{\prod_{r=s+1}^{t-1}(1+a(r))}{\prod_{t=0}^{T-1}(1+a(t)) - 1}, & 0 \leq s \leq t-1, \\[2ex] \dfrac{\prod_{r=0}^{t-1}(1+a(r)) \prod_{r=s+1}^{T-1}(1+a(r))}{\prod_{t=0}^{T-1}(1+a(t)) - 1}, & t \leq s \leq T-1. \end{cases} \tag{2.9}$$

Here we denote $\prod_{r=t}^{t-1}(1+a(r)) := 1$, $\prod_{t=T}^{T-1}(1+a(t)) := 1$.

A direct calculation shows that

$$A := \frac{1}{\prod_{t=0}^{T-1}(1+a(t)) - 1} \leq G(t,s) \leq \frac{\prod_{t=0}^{T-1}(1+a(t))}{\prod_{t=0}^{T-1}(1+a(t)) - 1} =: B,$$

and $\sigma = A/B < 1$. Hence, we have

$$\|x\| \leq B\sum_{t=0}^{T-1} x(s)g(s, y(s-\tau_1(s)), \dots, y(s-\tau_n(s))),$$

and

$$\min_{t\in Z[0,T-1]} x(t) \geq A \sum_{t=0}^{T-1} x(s)g(s,y(s-\tau_1(s)),\dots,y(s-\tau_n(s))).$$

From these we have

$$\min_{t\in Z[0,T-1]} x(t) \geq \frac{A}{B}\|x\| = \sigma\|x\|.$$

This completes the proof of Lemma 1.

Now we reach the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\Delta x(t) = \lambda x(t)[a(t) - g(t,x(t-\tau_1(t)),\dots,x(t-\tau_n(t))], \qquad x \in X. \tag{1.1$_\lambda$}$$

Let $x(t)$ be a positive T-periodic solution of $(1.1)_\lambda$. Then similar to the proof of Lemma 1, we have

$$\min_{t\in Z[0,T-1]} x(t) \geq \|x\| \left[\prod_{t=0}^{T-1}(1+\lambda a(t))\right]^{-1},$$

which implies $\min_{t\in Z[0,T-1]} x(t) \geq \sigma\|x\|$. Then we have the following lemma.

LEMMA 2. *Let $x(t)$ be a positive T-periodic solution of $(1.1)_\lambda$. Then*

$$\min_{t\in Z[0,T-1]} x(t) \geq \sigma\|x\|.$$

In order to prove the existence of a positive T-periodic solution of equation $(1.1)_\lambda$, we need the following lemma.

LEMMA 3. *Assume that (H_1) and (H_2) hold. Let $x(t)$ be a positive T-periodic solution of $(1.1)_\lambda$. Then*

$$\sigma\varepsilon < x(t) < \frac{H}{\sigma}, \qquad t \in Z[0,T-1].$$

PROOF. Since $\sum_{t=0}^{T-1}\Delta x(t) = 0$, then from $(1.1)_\lambda$, we have

$$\sum_{t=0}^{T-1} x(t)[a(t) - g(t,x(t-\tau_1(t)),\dots,x(t-\tau_n(t))] = 0. \tag{2.10}$$

If $\|x\| \geq H/\sigma$, then by Lemma 2,

$$\min_{t\in Z[0,T-1]} x(t) \geq \sigma\|x\| \geq H,$$

which means $\min_{t\in Z(-\infty,\infty)} x(t) \geq H$. So, we have

$$x(t-\tau_1(t)) \geq H,\dots,x(t-\tau_n(t)) \geq H, \qquad x \in X.$$

It follows from H_1 that

$$g(t,x(t-\tau_1(t)),\dots,x(t-\tau_n(t)) > a(t), \qquad t \in Z(-\infty,\infty).$$

Then equation (2.10) does not hold, which is a contradiction. This shows that $\|x\| < H/\sigma$.

Similarly, if $\min_{t\in Z[0,T-1]} x(t) \leq \sigma\varepsilon$, then by Lemma 2, $\|x\| \leq \varepsilon$. It follows from (H_2) that

$$g(t,x(t-\tau_1(t)),\dots,x(t-\tau_n(t)) < a(t), \qquad t \in Z(-\infty,\infty).$$

Then equation (2.10) does not hold either, which is a contradiction. This shows that $\min_{t\in Z[0,T-1]} x(t) > \sigma\varepsilon$. The lemma is thus proved.

Let

$$\Omega := \left\{ x \in X \mid \sigma\varepsilon < x(t) < \frac{H}{\sigma},\ t \in Z[0,T-1] \right\}.$$

It is clear that Ω verifies the requirement (a) in Theorem A. When $x \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R$, x is a constant with $x = \sigma\varepsilon$ or $x = H/\sigma$, and we have

$$QNx = \frac{1}{T}\sum_{t=0}^{T-1} x[a(t) - g(t,x,\dots,x)] \neq 0.$$

We define

$$\phi(\mu,x) = \mu\left[\frac{1}{2}\left(\sigma\varepsilon + \frac{H}{\sigma}\right) - x\right] + (1-\mu)QNx, \qquad 0 \le \mu \le 1.$$

Since from (H_1) and (H_2), we know that $\phi(\mu,x) \neq 0$ when $x \in \partial\Omega \cap R$; then we have $\phi(\mu,x) \neq 0$. In view of the homotopic invariant property of topological degree, from (H_1) and (H_2), it is easy to see that

$$\begin{aligned}\deg\{JQNx, \Omega\cap R, 0\} &= \deg\left\{\left[\frac{1}{2}\left(\sigma\varepsilon + \frac{H}{\sigma}\right) - x\right], \Omega\cap R, 0\right\}\\ &\neq 0\end{aligned}$$

(noting that $J = I$). By now we have proved that Ω verifies all the requirements in Theorem A. Hence, equation (1.1) has at least one positive T-periodic solution in $\bar{\Omega}$.

The proof of Theorem 1 is complete.

In a similar way as the proof of Theorem 1, we have the following theorem.

THEOREM 2. *Suppose the following.*

(H_1) *There exists $H > 0$ such that*

$$g(t,u_1,u_2,\dots,u_n) < a(t), \qquad \text{for } t \in Z[0,T-1],$$

when $u_i \ge H$, $i = 1,\dots,n$.

(H_2) *There exists $\varepsilon > 0$ ($\varepsilon < H$) such that*

$$g(t,u_1,u_2,\dots,u_n) > a(t), \qquad \text{for } t \in Z[0,T-1],$$

when $0 < u_i \le \varepsilon$, $i = 1,\dots,n$.

Then equation (1.1) has at least one positive periodic solution of period T.

3. EXAMPLES

In this section, we apply the main results obtained in the previous section to study some examples which have some biological background.

From Theorem 2.1, we have the following corollaries.

COROLLARY 1. *Equation (1.2) has at least one positive periodic solution of period T.*

COROLLARY 2. *Equation (1.3) has at least one positive periodic solution of period T.*

COROLLARY 3. *Equation (1.4) has at least one positive periodic solution of period T.*

COROLLARY 4. *Assume that* $\sum_{i=1}^{n} a_i(t)/c_i(t) > 1$. *Then equation (1.5) has at least one positive periodic solution of period* T.

Corollaries 1–3 can be checked easily.

For Corollary 4, since

$$\lim_{u_1,\dots,u_n\to\infty} a(t)\sum_{i=1}^{n} \frac{a_i(t)u_i}{1+c_i(t)u_i} = a(t)\sum_{i=1}^{n}\frac{a_i(t)}{c_i(t)} > a(t), \qquad t\in Z[0,T-1],$$

and

$$\lim_{u_1,\dots,u_n\to 0^+} a(t)\sum_{i=1}^{n} \frac{a_i(t)u_i}{1+c_i(t)u_i} = 0, \qquad t\in Z[0,T-1],$$

all the assumptions in Theorem 1 are satisfied, and the conclusion follows.

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